

# Symplectomorphism Groups and Almost Complex Structures

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**Abstract** This paper studies groups of symplectomorphisms of ruled surfaces  $M$  for symplectic forms with varying cohomology class. This cohomology class is characterised by the ratio  $\mu$  of the size of the base to that of the fiber. By considering appropriate spaces of almost complex structures, we investigate how the topological type of these groups changes as  $\mu$  increases. If the base is a sphere, this changes precisely when  $\mu$  passes an integer, and, for general bases, it stabilizes as  $\mu \rightarrow \infty$ . Our results extend and make more precise some of the conclusions of Abreu–McDuff concerning the rational homotopy type of these groups for rational ruled surfaces.

## 1 Introduction

One of the interesting facts of symplectic geometry is that symplectic manifolds admit a large family of symplectomorphisms, i.e. diffeomorphisms that preserve the symplectic structure. Indeed, every function  $H$  on a closed symplectic manifold  $(M, \omega)$  generates a flow  $\phi_t^H, t \in \mathbb{R}$ , consisting of symplectomorphisms. Thus the group  $\text{Symp}(M, \omega)$  of all symplectomorphisms of  $(M, \omega)$  is always infinite dimensional. On the other hand, in some special cases the homotopy type of this group can be calculated and it turns out to be not too large. For example, although nothing at all is known about the group of compactly supported diffeomorphisms of  $\mathbb{R}^4$  — it is not even known whether it is connected — Gromov showed in 1985 (see [4]) that the group of compactly supported symplectomorphisms of  $\mathbb{R}^4$  with its standard symplectic structure is contractible. He also showed that the group of symplectomorphisms of  $S^2 \times S^2$ , with the product symplectic form in which both spheres

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have the same size, has the homotopy type of a Lie group: in fact it is homotopic to the semi-direct product of  $\mathrm{SO}(3) \times \mathrm{SO}(3)$  with  $\mathbb{Z}/2\mathbb{Z}$ .

More recently, Abreu [1] showed that when one sphere factor is larger than the other the group of symplectomorphisms is no longer homotopy equivalent to a Lie group, because it does not have the right kind of rational homotopy type. This rational homotopy type was calculated in Abreu–McDuff [2], and was found to change precisely when the ratio  $\mu$  of the size of the larger to the smaller sphere passes an integer value.

The current paper extends this last result to the actual rather than the rational homotopy type. Our arguments are more general than before in that in principle they apply to ruled surfaces over Riemann surfaces of arbitrary genus  $g$ . However, in order to get sharp results when  $g > 0$  we would need detailed information about the question of which homology classes are represented by embedded  $J$ -holomorphic curves for *arbitrary* (and hence non generic) tame almost complex structures  $J$ . This is a rather delicate matter that will be explored in a later paper.

The main idea in this paper is to consider the fibration

$$\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M) \rightarrow \mathrm{Diff}_0(M) \rightarrow \mathcal{S}_{[\omega]}$$

coming from the action of the identity component  $\mathrm{Diff}_0(M)$  of the group of diffeomorphisms on the space  $\mathcal{S}_{[\omega]}$  of all symplectic structures on  $M$  that are isotopic to  $\omega$ , and then to center the arguments on the spaces  $\mathcal{S}_{[\omega]}$  rather than on the groups  $\mathrm{Symp}(M, \omega)$ . This point of view was inspired by Kronheimer’s construction in [7] of families of symplectic forms that represent elements in  $\pi_*(\mathcal{S}_{[\omega]})$  with nonzero image under the boundary map

$$\partial : \pi_*(\mathcal{S}_{[\omega]}) \rightarrow \pi_{*-1}(\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M)).$$

This approach allows us to recover some of the main results in [2], though it does not lead to a complete calculation of the homotopy type of  $\mathrm{Symp}(M, \omega)$ . So far, the only such calculation besides Gromov’s is by Anjos [3], who recently found the homotopy type of the symplectomorphism group of  $S^2 \times S^2$  when the ratio  $\mu$  of the sizes of the spheres lies in the interval  $(1, 2]$ .

An essential tool in the argument is the existence of many  $J$ -holomorphic curves in ruled surfaces, which is a reflection of the fact that they have many nonzero Seiberg–Witten invariants. It is not at all clear what can be said about the topology of  $\mathrm{Symp}(M, \omega)$  for general 4-manifolds that typically have few such curves. Also, nothing is known in higher dimensions. For example, it is not known whether the group of compactly supported symplectomorphisms of  $\mathbb{R}^{2n}$  is contractible when  $n > 2$ .

## 1.1 Statement of main results

Ruled surfaces are compact smooth 4-manifolds  $M$  that fiber over a Riemann surface  $\Sigma = \Sigma_g$  of genus  $g$  with fiber  $S^2$ . There are two topological types, the product  $\Sigma \times S^2$  and the total space  $M_\Sigma$  of a nontrivial fibration over  $\Sigma$ . As in [2], the two cases are analogous. For simplicity, we will restrict ourselves here to the case of the trivial fibration.

It is known from the work of Taubes, Li–Liu and Lalonde–McDuff (for detailed references see [9]) that every symplectic form  $\omega$  on  $\Sigma \times S^2$  is diffeomorphic to a scalar multiple of one of the standard forms  $\omega_\mu$ , where

$$\omega_\mu = \mu\sigma_\Sigma + \sigma_{S^2}, \quad \mu > 0.$$

(Here  $\sigma_Y$  denotes an area form on the Riemann surface  $Y$  with total area 1.) We denote by  $G_\mu = G_\mu^g$  the subgroup  $\text{Symp}(M, \omega_\mu) \cap \text{Diff}_0(M)$  of the group of symplectomorphisms of  $(M, \omega_\mu)$ . When  $g > 0$   $\mu$  ranges over all positive numbers. However, when  $g = 0$  there is an extra symmetry: interchanging the two spheres gives an isomorphism  $G_\mu^0 \cong G_{1/\mu}^0$ . It is important for our arguments that the base sphere be at least as large as the fiber: see Lemma 2.2. Hence in this case we take  $\mu$  in the range  $[1, \infty)$ .

The groups  $G_\mu^g$  were first studied by Gromov [4] in the case when  $g = 0$ . He showed that when  $\mu = 1$ , i.e. when both spheres have the same size,  $G_1^0$  is connected and deformation retracts to the Lie group  $\text{SO}(3) \times \text{SO}(3)$ . He also pointed out that as soon as  $\mu$  gets bigger than 1, a new element of infinite order appears in the fundamental group  $\pi_1(G_\mu^0)$ . The key idea in his proof was to look at the action of  $G_\mu^0$  on the contractible space  $\mathcal{J}_\mu$  of  $\omega_\mu$ -compatible almost complex structures.

These ideas were taken much further by Abreu [1] and Abreu–McDuff [2] and led to a calculation of the rational homotopy type of both  $G_\mu^0$  and  $BG_\mu^0$  for all  $\mu$ . Three of these results are relevant here.

**Proposition 1.1** *As  $\mu \rightarrow \infty$ , the groups  $G_\mu^g$  tend to a limit  $G_\infty^g$  that has the homotopy type of the identity component  $\mathcal{D}_0^g$  of the group of fiberwise diffeomorphisms of  $M = \Sigma_g \times S^2$ .*

Here  $\mathcal{D}_0 = \mathcal{D}_0^g$  is the identity component of the group  $\mathcal{D} = \mathcal{D}^g$  of all diffeomorphisms  $\phi$  that fit into the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\phi} & M \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{\phi'} & \Sigma, \end{array}$$

where the vertical arrows are projection to the first factor.  $\mathcal{D}_0^g$  can also be described as the set of all diffeomorphisms that are isotopic to the identity and preserve the degenerate form  $\sigma_\Sigma$ . The calculation of the limit of the  $G_\mu^g$  is straightforward: almost the hardest part is the definition of the maps  $G_\mu^g \rightarrow G_{\mu+\varepsilon}^g$ . In §2.1 below we give an indirect but conceptually simple definition of these maps.

Observe also that  $\mathcal{D}_0^0$  is homotopy equivalent to  $\text{SO}(3) \times \text{SO}(3) \times \Omega^2(S^3)$ , where  $\Omega^2(S^3)$  denotes the double loop space of  $S^3$ : see §5.1. Gromov’s new element in  $\pi_1(G_\mu^0)$  maps to the generator of  $\pi_1(\Omega^2(S^3))$ . Geometrically, it can be represented by the “rotation about the diagonal and anti-diagonal”, i.e. by the family of maps

$$S^1 \times S^2 \times S^2 \rightarrow S^2 \times S^2 : (t, z, w) \mapsto (z, R_{z,t}(w)),$$

where  $R_{z,t}$  is the rotation of the fiber sphere  $S^2$  through the angle  $2\pi t$  about the axis through the point  $z \in S^2$ .

Here is the second main result from [2].

**Proposition 1.2** *When  $\ell < \mu \leq \ell + 1$  for some integer  $\ell \geq 1$ ,*

$$H^*(G_\mu^0, \mathbb{Q}) = \Lambda(t, x, y) \otimes \mathbb{Q}[w_\ell],$$

where  $\Lambda(t, x, y)$  is an exterior algebra over  $\mathbb{Q}$  with generators  $t$  of degree 1, and  $x, y$  of degree 3 and  $\mathbb{Q}[w_\ell]$  is the polynomial algebra on a generator  $w_\ell$  of degree 4 $\ell$ .

In the above statement, the generators  $x, y$  come from  $H^*(\mathrm{SO}(3) \times \mathrm{SO}(3))$  and  $t$  corresponds to Gromov's element in  $\pi_1(G_\mu^0)$ ,  $\mu > 1$ . Thus the subalgebra  $\Lambda(t, x, y)$  is the pullback of  $H^*(\mathcal{D}_0^0, \mathbb{Q})$  under the map  $G_\mu \rightarrow \mathcal{D}_0^0$ . The other generator  $w_\ell$  is fragile, in the sense that it disappears (i.e. becomes null cohomologous) when  $\mu$  increases.

Thus the map  $G_\mu^0 \rightarrow \mathcal{D}_0^0$  is not a homotopy equivalence for any  $\mu$ . However, as we shall see in Corollary 5.5 the next statement implies that it induces a surjection on rational homotopy when  $\mu > 1$ .

**Proposition 1.3** *When  $\mu > 1$ ,  $G_\mu^0$  is connected and  $\pi_1(G_\mu^0) = \pi_1(\mathcal{D}_0^0) = \mathbb{Z} + \mathbb{Z}/2\mathbb{Z} + \mathbb{Z}/2\mathbb{Z}$ .*

In this paper we give a simplified proof of Proposition 1.1 above and a new proof of the following result about the groups  $G_\mu^0$ . Part (i) of the theorem below sharpens one aspect of Proposition 1.2, while part (ii) is somewhat weaker.

**Theorem 1.4** *(i) The homotopy type of  $G_\mu^0$  is constant on all intervals  $(\ell, \ell + 1]$  with  $\ell \geq 1$ . Moreover, as  $\mu$  passes the integer  $\ell + 1$ ,  $\ell \geq 0$ , the groups  $\pi_i(G_\mu^0)$ ,  $i \leq 4\ell - 1$ , do not change.*

*(ii) There is an element  $\rho_\ell \in \pi_{4\ell}(G_\mu^0) \otimes \mathbb{Q}$  when  $\ell < \mu \leq \ell + 1$  that vanishes for  $\mu > \ell + 1$ .*

Here, the elements  $\rho_\ell$  are dual to the classes  $w_\ell \in H^{4\ell}(G_\mu^0, \mathbb{Q})$  of Proposition 1.2. This result is proved by methods that should in principle work for base manifolds  $\Sigma$  of any genus. (In contrast the proofs in [2] use ideas that definitely do not hold when  $g > 0$ : see Remark 5.7.) A natural conjecture would be that the analog of (i) should hold for all  $g$ , perhaps with the proviso that  $\ell \geq g$ . Statement (ii) should also hold with the dimension of  $\rho_\ell$  being taken as  $4\ell + 2g$ . As we shall see in §2.2 below, what we would need to extend our results in this way is a better knowledge of which classes in  $H_2(M)$  can be represented by embedded  $J$ -holomorphic curves. We will content ourselves here with stating a few easy partial results. The first result concerns the case  $g = 1$  and the second is an analog of Proposition 1.3.

**Proposition 1.5** *(i) The homotopy type of the groups  $G_\mu^1$  is constant for  $0 < \mu \leq 1$ . Moreover, the map  $G_\mu^1 \rightarrow \mathcal{D}_0^1$  gives rise to isomorphisms on  $\pi_i$ ,  $i = 0, 1$ , for all  $\mu > 0$ , and isomorphisms on  $\pi_i$ ,  $i = 2, \dots, 5$ , for all  $\mu > 3/2$ .*

*(ii) There is a nonzero element  $\rho \in \pi_2(G_1^1)$  that vanishes in  $\pi_2(G_\mu^1)$  when  $\mu > 1$ .*

**Proposition 1.6** *(i) If  $g = 2k$  or  $2k + 1$  and  $\mu > k$  then the map  $G_\mu^g \rightarrow \mathcal{D}_0^g$  gives rise to an isomorphism on  $\pi_i$  for  $i \leq 2g - 1$ . In particular,  $G_\mu^g$  is connected whenever  $\mu > k$ .*

*(ii) When  $g > 0$ , the induced map  $G_\mu^g \rightarrow \mathcal{D}_0^g$  induces a surjection on rational homotopy groups for all  $\mu > 0$ .*

### Further Questions

- As noted in Proposition 1.5 above, the homotopy type of  $G_\mu^1$  is constant for  $0 < \mu \leq 1$ . It would be interesting to figure out its structure. In the genus 0 case, it was shown in [2] that there is a family of Lie subgroups  $K_k$  of  $\mathrm{Diff}_0(M)$  such that the whole rational homotopy

type of  $G_\mu^0$  is generated by the  $K_k$  for  $k < \mu$ .<sup>1</sup> In particular, the new fragile elements  $\rho_k$  are higher Whitehead (or Samelson) products of elements in  $\pi_*(K_k)$ . It is not clear if there is an analogous result when  $g = 1$ . In particular, it would be nice to find a specific representative of the element  $\rho$  in  $\pi_2(G_1^1)$ .

- It seems very likely that the group  $G_\mu^g$  is always connected. Equivalently, it is likely that the map

$$\pi_0(\mathrm{Symp}(M, \omega_\mu)) \rightarrow \pi_0(\mathcal{D}^g),$$

which can be shown to be a surjection as in the proof of Proposition 1.6 (ii), is in fact an isomorphism.

- Another direction in which one might speculate concerns the relation of the groups  $G_\mu^g$  to the full group of diffeomorphisms. For example, consider the case  $M = S^2 \times S^2$ . Then  $\mathrm{Diff}_0(M)$  contains two copies of  $\mathcal{D}_0^0$  which we might call  $\mathcal{D}_{left}$  and  $\mathcal{D}_{right}$  corresponding to the two projections  $M \rightarrow S^2$ . They intersect in  $\mathrm{SO}(3) \times \mathrm{SO}(3) = G_1^0$ . It would be interesting to know how much of the topology of  $\mathrm{Diff}_0(S^2 \times S^2)$  is generated by these subgroups. Could it possibly be true that, just as the groups  $K_k$  generate the rational homotopy of  $G_\mu^0$ , these two groups generate the rational homotopy of  $\mathrm{Diff}_0(S^2 \times S^2)$ ? In particular, what is the Samelson product of the two elements  $\tau_{left}, \tau_{right}$  of  $\pi_1(\mathrm{Diff}_0(S^2 \times S^2))$  given by the generator  $\tau$  of  $\pi_1(\mathcal{D}_0^0)$ ? There are similar questions when  $g > 0$ . However there is less topology to play with, since the groups  $\mathrm{Diff}_0(\Sigma_g)$  are contractible when  $g > 1$ .

**Organization of the paper** In §2 we first show how to define and calculate the limit  $G_\infty$ , and then describe the main points in the proofs of Theorem 1.4, Proposition 1.5 and Proposition 1.6 (i). The next two sections discuss some needed techniques: symplectic inflation in §3 and the construction of embedded curves in §4. Finally, in §5 we study the topology of the fiberwise groups  $\mathcal{D}_g^0$  and prove Proposition 1.6 (ii). This is elementary except for the statement that  $\pi_1(G_\mu^g)$  maps onto  $\pi_1(\mathcal{D}_0^g)$  when  $\mu \leq 1$  and  $g > 1$ .

## 2 Outline of proofs

In this section we define and calculate the limit  $G_\infty$  and then show how to understand the relationship between the different spaces  $\mathcal{A}_\mu$ .

### 2.1 The basic idea

There is no very direct map  $G_\mu \rightarrow G_{\mu+\varepsilon}$  for  $\varepsilon > 0$ , and the proof of Proposition 1.1 given in [2] is rather clumsy. This section gives a streamlined version of the argument that is based on looking at the fibration

$$G_\mu \rightarrow \mathrm{Diff}_0(M) \rightarrow \mathcal{S}_\mu.$$

Here  $\mathrm{Diff}_0(M)$  is the identity component of the group of diffeomorphisms of  $M$  and  $\mathcal{S}_\mu$  is the space of all symplectic forms on  $M$  in the cohomology class  $[\omega_\mu]$  that are isotopic to  $\omega_\mu$ .

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<sup>1</sup>By Anjos [3], this statement remains true for the integral homotopy type of  $G_\mu^0$ ,  $1 < \mu \leq 2$ .

By Moser's theorem the group  $\text{Diff}_0(M)$  acts transitively on  $\mathcal{S}_\mu$  via an action that we will write as

$$\phi \cdot \omega = \phi_*(\omega) = (\phi^{-1})^*\omega,$$

so that  $\mathcal{S}_\mu$  is simply the homogeneous space  $\text{Diff}_0(M)/G_\mu$ . Let  $\mathcal{A}_\mu$  denote the space of almost complex structures that are tamed by some form in  $\mathcal{S}_\mu$ .<sup>2</sup>

**Lemma 2.1**  $\mathcal{S}_\mu$  is homotopy equivalent to  $\mathcal{A}_\mu$ .

**Proof:** Let  $\mathcal{X}_\mu$  be the space of pairs

$$\mathcal{X}_\mu = \{(\omega, J) \in \mathcal{S}_\mu \times \mathcal{A}_\mu : \omega \text{ tames } J\}.$$

Then both projection maps  $\mathcal{X}_\mu \rightarrow \mathcal{A}_\mu, \mathcal{X}_\mu \rightarrow \mathcal{S}_\mu$  are fibrations with contractible fibers (see [16] Ch 2.5), and so are homotopy equivalences.  $\square$

**Lemma 2.2** If  $g > 0$  then  $\mathcal{A}_\mu \subset \mathcal{A}_{\mu+\varepsilon}$  for all  $\mu, \varepsilon > 0$ . When  $g = 0$  this holds provided  $\mu \geq 1$ .

**Sketch of proof:** It is well known that for each  $J \in \mathcal{A}_\mu$ ,  $M$  admits a foliation whose leaves are  $J$ -holomorphic spheres representing the fiber class  $F = [pt \times S^2]$ . For the sake of completeness, we sketch a proof in Lemma 4.1 below. Observe that this is where we use the fact that  $\mu \geq 1$  when  $\Sigma = S^2$ : if the base is smaller than the fiber, the fiber class can degenerate so that  $\pi_J$  may not exist.

It follows that there is a projection  $\pi_J : M \rightarrow \Sigma$  onto the leaf space  $\Sigma$  of this foliation. We show in §3 that if  $\sigma$  is any area form on  $\Sigma$  then its pullback  $\pi_J^*(\sigma)$  is  $J$ -semi-tame, i.e.

$$\pi_J^*(\sigma)(v, Jv) \geq 0, \quad v \in TM.$$

Granted this, if  $\omega \in \mathcal{S}_\mu$  tames  $J$ , then  $\omega + \kappa\pi_J^*(\sigma)$  also tames  $J$  for all  $\kappa > 0$ . The result is then immediate.  $\square$

**Corollary 2.3** For any  $\mu, \varepsilon, \varepsilon' > 0$  there are maps  $\mathcal{S}_\mu \rightarrow \mathcal{S}_{\mu+\varepsilon}$  and  $G_\mu \rightarrow G_{\mu+\varepsilon}$  that are well defined up to homotopy and make the following diagrams homotopy commute:

$$(a) \quad \begin{array}{ccccccc} G_\mu & \rightarrow & \text{Diff}_0(M) & \rightarrow & \mathcal{S}_\mu \\ \downarrow & & \downarrow = & & \downarrow \\ G_{\mu+\varepsilon} & \rightarrow & \text{Diff}_0(M) & \rightarrow & \mathcal{S}_{\mu+\varepsilon}, \end{array}$$
  

$$(b) \quad \begin{array}{ccc} G_\mu & \rightarrow & G_{\mu+\varepsilon} \\ & \searrow & \downarrow \\ & & G_{\mu+\varepsilon+\varepsilon'} \end{array} .$$

**Proof:** The maps  $\mathcal{S}_\mu \rightarrow \mathcal{S}_{\mu+\varepsilon}$  are induced from the inclusions  $\mathcal{A}_\mu \subset \mathcal{A}_{\mu+\varepsilon}$  using the homotopy equivalences  $\mathcal{S}_\mu \simeq \mathcal{A}_\mu$  in Lemma 2.1 above. Since  $G_\mu$  is the fiber of the map  $\text{Diff}_0(M) \rightarrow \mathcal{S}_\mu$ , there are induced maps  $G_\mu \rightarrow G_{\mu+\varepsilon}$  making diagram (a) homotopy commute. The rest is obvious.  $\square$

<sup>2</sup>Recall that  $\omega$  is said to tame  $J$  if  $\omega(v, Jv) > 0$  for all  $v \neq 0$ .

This corollary illustrates the essential feature of our approach. Statements that are true only up to homotopy on the level of the groups  $G_\mu$  are true on the nose for the spaces  $\mathcal{A}_\mu$ . The discussion below shows that it is easy to understand what happens to the groups  $G_\mu$  as  $\mu$  increases. Much of the rest of the paper is devoted to understanding (on the level of the spaces  $\mathcal{A}_\mu$ ) what happens as  $\mu$  decreases. For this we use the Lalonde–McDuff technique of symplectic inflation.

### Proof of Proposition 1.1.

We first show that we can understand the limit  $G_\infty = \lim G_\mu$  by studying the space  $\cup_\mu \mathcal{A}_\mu$ . Let  $J_{split}$  be the standard product almost complex structure on  $M$ . Because  $J_{split}$  is tamed by  $\omega_\mu$  the map  $\text{Diff}_0(M) \rightarrow \mathcal{S}_\mu$  lifts to

$$\text{Diff}_0(M) \rightarrow \mathcal{X}_\mu : \quad \phi \mapsto (\phi_*(\omega_\mu), \phi_*(J_{split})).$$

Composing with the projection to  $\mathcal{A}_\mu$  we get a map

$$\text{Diff}_0(M) \rightarrow \mathcal{A}_\mu : \quad \phi \mapsto \phi_*(J_{split})$$

that is not a fibration but has homotopy fiber  $G_\mu$ . Since  $\mathcal{A}_\mu$  is an open subset of  $\mathcal{A}_{\mu+\varepsilon}$  for all  $\varepsilon > 0$ , the homotopy limit  $\lim_\mu \mathcal{A}_\mu$  of the spaces  $\mathcal{A}_\mu$  is homotopy equivalent to the union  $\mathcal{A}_\infty = \cup_\mu \mathcal{A}_\mu$ . Hence  $G_\infty$ , which is defined to be the homotopy limit of the  $G_\mu$ , is homotopy equivalent to the homotopy fiber of the map  $\text{Diff}_0(M) \rightarrow \mathcal{A}_\infty$ .

To understand  $\mathcal{A}_\infty$  we proceed as follows. Let  $\text{Fol}$  be the space of all smooth foliations of  $\Sigma \times S^2$  by spheres in the fiber class  $F = [pt \times S^2]$ . Since  $S^2$  is compact and simply connected, each leaf of this foliation has trivial holonomy and hence has a neighborhood that is diffeomorphic to the product  $D^2 \times S^2$  equipped with the trivial foliation with leaves  $pt \times S^2$ . It follows that  $\text{Diff}(M)$  acts transitively on  $\text{Fol}$  via the map  $\phi \mapsto \phi(\mathcal{F}_{split})$ , where  $\mathcal{F}_{split}$  is the flat foliation by the spheres  $pt \times S^2$ . Similarly,  $\text{Diff}_0(M)$  acts transitively on the connected component  $\text{Fol}_0$  of  $\text{Fol}$  that contains  $\mathcal{F}_{split}$ . Hence there is a fibration sequence

$$\mathcal{D} \cap \text{Diff}_0(M) \rightarrow \text{Diff}_0(M) \rightarrow \text{Fol}_0.$$

It is not hard to see that the group  $\mathcal{D} \cap \text{Diff}_0(M)$  is connected, and so equal to  $\mathcal{D}_0$ : see Corollary 5.2.

Next, observe that there is a map  $\mathcal{A}_\infty \rightarrow \text{Fol}_0$  given by taking  $J$  to the foliation of  $M$  by  $J$ -spheres in class  $F$ . Standard arguments (see for example [16] Ch 2.5) show that this map is a fibration with contractible fibers. Hence it is a homotopy equivalence. Moreover, it fits into the commutative diagram:

$$\begin{array}{ccc} \text{Diff}_0(M) & \rightarrow & \mathcal{A}_\infty \\ \downarrow & & \downarrow \\ \text{Diff}_0(M) & \rightarrow & \text{Fol}_0, \end{array}$$

where the map  $\text{Diff}_0(M) \rightarrow \mathcal{A}_\infty$  is given as above by the action  $\phi \mapsto \phi_*(J_{split})$ . Hence there is an induced homotopy equivalence from the homotopy fiber  $G_\infty$  of the top row to the fiber  $\mathcal{D}_0$  of the second.  $\square$

**Remark 2.4** Implicit in the above argument is the following description of the map  $G_\infty \rightarrow \mathcal{D}_0$ . Let  $\mathcal{J}_\mu$  denote the space of all almost complex structures tamed by  $\omega_\mu$ . Since the image

of the group  $G_\mu$  under the map  $\text{Diff}_0(M) \rightarrow \mathcal{A}_\mu$  is contained in  $\mathcal{J}_\mu$  there is a commutative diagram

$$\begin{array}{ccc} & \mathcal{D}_0 & \\ & \downarrow & \\ G_\mu & \xrightarrow{\iota} & \text{Diff}_0(M) \\ \downarrow & & \downarrow \\ \mathcal{J}_\mu & \longrightarrow & \text{Fol.} \end{array}$$

Because  $\mathcal{J}_\mu$  is contractible, the inclusion  $\iota : G_\mu \rightarrow \text{Diff}_0(M)$  lifts to a map  $\tilde{\iota} : G_\mu \rightarrow \mathcal{D}_0$ . Now take the limit to get  $G_\infty \rightarrow \mathcal{D}_0$ .

## 2.2 The stratification of $\mathcal{A}_\mu$

The following definition is the key to understanding the relation between the  $\mathcal{A}_\mu$  as  $\mu$  decreases.

**Definition 2.5** Given  $\ell > 0$ , let  $\mathcal{A}_{\mu,\ell}$  be the subset of  $J \in \mathcal{A}_\mu$  consisting of elements that admit a  $J$ -holomorphic curve in class  $A - \ell F$ . Here  $F$  is the fiber class as before, and  $A = [\Sigma \times pt]$ . Further, define

$$\mathcal{A}_{\mu,0} = \mathcal{A}_\mu - \cup_{\ell>0} \mathcal{A}_{\mu,\ell}.$$

Since the fiber class  $F$  is always represented and since distinct  $J$ -holomorphic curves always intersect positively,<sup>3</sup> the class  $pA + qF$  has a  $J$ -holomorphic representative only if

$$p = (pA + qF) \cdot F \geq 0.$$

Thus  $\mathcal{A}_{\mu,0}$  can be characterized as the set of  $J$  for which the only classes with  $J$ -holomorphic representatives are  $pA + qF$  with  $p, q \geq 0$ . A similar argument shows that the sets  $\mathcal{A}_{\mu,\ell}$  are disjoint for  $\ell \geq 0$ . Note also that  $\mathcal{A}_{\mu,\ell}$  is nonempty only if  $\ell < \mu$ : if  $J$  admits a  $J$ -holomorphic curve in class  $A - \ell F$ , any form  $\omega \in \mathcal{S}_\mu$  that tames  $J$  must be positive on this curve. Hence  $\omega(A - \ell F) = \mu - \ell > 0$ .

The next lemma shows that the subsets  $\mathcal{A}_{\mu,\ell}$  are well behaved.

**Lemma 2.6** For all  $\ell > 0$ ,  $\mathcal{A}_{\mu,\ell}$  is a Fréchet suborbifold of  $\mathcal{A}_\mu$  of codimension  $n(\ell) = 4\ell - 2 + 2g$ .

**Proof:** This is proved in the genus 0 case by Abreu [1], and the proof for general  $g$  is similar. (See also [14].) The idea is the following. Let  $\mathcal{M}$  denote the moduli space of all triples  $(u, j, J)$  where  $J \in \mathcal{A}_\mu$ ,  $(\Sigma, j)$  is a marked Riemann surface (so  $j$  is in Teichmüller space) and  $u : (\Sigma, j) \rightarrow M$  is a  $J$ -holomorphic curve in class  $A - \ell F$ . By standard theory  $\mathcal{M}$  is a Fréchet manifold. The forgetful map

$$\mathcal{M} \rightarrow \mathcal{A}_\mu, \quad (u, J) \mapsto J,$$

has image  $\mathcal{A}_{\mu,\ell}$ . Moreover its fiber at  $J$  is the fiber of the projection from Teichmüller space to the moduli space of Riemann surfaces of genus  $g$ . Hence it varies by at most a finite group. This gives  $\mathcal{A}_{\mu,\ell}$  the structure of a Fréchet orbifold.

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<sup>3</sup>See [8] for an outline of the main points of the theory of  $J$ -holomorphic curves, and Sikorav [17] for detailed proofs of positivity of intersections.

It remains to check that it has finite codimension in  $\mathcal{A}_\mu$ . To see this, note that the curve  $C = \text{Im } u$  is embedded since each fiber intersects it exactly once, positively. (By positivity of intersections, if  $u$  were singular at  $z \in \Sigma$ , then the intersection number of  $C$  with the fiber through  $u(z)$  would have to be  $> 1$ .) Moreover  $c_1(A - \ell F) < 0$ . Hofer–Lizan–Sikorav showed in [5] that in this case the linearization

$$Du : C^\infty(u^*TM) \rightarrow \Omega_J^{0,1}(u^*TM)$$

of the generalized Cauchy–Riemann operator  $\partial_J$  at  $u$  has cokernel of constant rank equal to  $n(\ell)$ . Moreover, Abreu’s arguments show that this cokernel can be identified with the normal space to  $\mathcal{A}_{\mu,\ell}$  at  $J$  as follows. Since  $\mathcal{A}_\mu$  is an open subset in the space of all almost complex structures on  $M$ , the tangent space  $T_J\mathcal{A}_\mu$  consists of all smooth sections  $Y$  of the bundle  $\text{End}_J(TM)$  of endomorphisms of  $TM$  that anticommute with  $J$ . The claim is that the restriction map

$$\text{End}_J(TM) \rightarrow \Omega_J^{0,1}(u^*(TM)) : Y \mapsto Y \circ du \circ j$$

identifies the fiber of the normal bundle at  $J$  with the cokernel of  $Du$ . By standard arguments (see for example Proposition 3.4.1 in [15]) we can restrict attention here to sections  $Y$  with support in some open set that intersects the curve  $C$ .  $\square$

The following key result is proved in §3 using the technique of symplectic inflation from [9].

**Proposition 2.7** *Let  $J \in \mathcal{A}_\mu$ . If the class  $pA + qF$  is represented by an embedded  $J$ -holomorphic curve  $C$  for some  $p > 0, q \geq 0$ , then  $J \in \mathcal{A}_{\varepsilon+q/p}$  for all  $\varepsilon > 0$ .*

The idea of the proof is to move the cohomology class of the taming form  $\omega$  for  $J$  towards the Poincaré dual of  $[C]$  by adding to  $\omega$  a large multiple of a form  $\tau_C$  that represents this class: see § 3. It follows that if we can find classes of small “slope”  $q/p$  that have embedded  $J$ -holomorphic representatives, we can decrease the parameter  $\mu$  of the set  $\mathcal{A}_\mu$  containing  $J$ . Note that if  $J \in \mathcal{A}_{\mu,\ell}$  and  $pA + qF$  has an embedded  $J$ -representative, then positivity of intersections implies that

$$(A - \ell F) \cdot (pA + qF) \geq 0, \quad \text{i.e. } \frac{q}{p} \geq \ell.$$

Thus if  $J \in \mathcal{A}_{\mu,\ell}$  we cannot hope to represent a class of smaller slope than  $A + \ell F$ .

The following result is proved in §4.

**Lemma 2.8** *If the base has genus  $g = 0$  and  $J \in \mathcal{A}_{\mu,\ell}$  then the class  $A + \ell F$  always has an embedded  $J$ -representative.*

**Corollary 2.9** *When  $g = 0$ ,  $\mathcal{A}_{\mu,\ell} = \mathcal{A}_{\ell+\varepsilon,\ell}$  for all  $\varepsilon > 0$ . Thus the sets  $\mathcal{A}_\mu = \cup_{\ell < \mu} \mathcal{A}_{\mu,\ell}$  are constant on all intervals  $k < \mu \leq k + 1$ .*

**Proof of Theorem 1.4 (i)**

We must show that the homotopy type of the groups  $G_\mu^0$  is constant for  $\mu \in (\ell, \ell + 1]$ . Because of the fibration

$$G_\mu \rightarrow \text{Diff}_0(M) \rightarrow \mathcal{S}_\mu,$$

it suffices to prove this for  $\mathcal{S}_\mu$  and hence, by Lemma 2.1, for  $\mathcal{A}_\mu$ . But this holds by the preceding corollary.

To prove the second statement, use Lemma 2.6 which says that when  $\mu$  passes  $\ell + 1$  the topology of  $\mathcal{A}_\mu$  changes by the addition of a stratum of codimension  $n(\ell) = 4\ell + 2$ . Hence its homotopy groups  $\pi_i$  for  $i \leq 4\ell$  do not change. Thus the groups  $\pi_i(G_\mu^0)$  for  $i \leq 4\ell - 1$ , do not change.  $\square$

It is not clear whether Lemma 2.8 continues to hold for arbitrary  $g$ . The problem is that we have to understand the  $J$ -holomorphic curves for *every*  $J$  not just the generic ones, or even just the integrable ones. In §4 we exhibit the existence of enough embedded  $J$ -holomorphic curves to prove:

**Lemma 2.10** *If  $g = 2k$  or  $2k + 1$  for some  $k \geq 0$ , the space  $A_{\mu,0}^g$  is constant for  $\mu > k$ . Moreover, the space  $A_{\mu,1}^1$  is constant for  $\mu > 3/2$ .*

**Corollary 2.11** *Part (i) of Propositions 1.5 and 1.6 hold.*

**Proof:** Because  $\mathcal{A}_\mu^g = \mathcal{A}_{\mu,0}^g$  when  $\mu \leq 1$ , Lemma 2.10 implies that  $\mathcal{A}_\mu^1$  is constant for  $\mu \in (0, 1]$ . Hence the first statement of Proposition 1.5 (i) holds by the arguments given above. Moreover, if  $\mu > 3/2$  and  $\varepsilon > 0$  the difference  $\mathcal{A}_{\mu+\varepsilon}^1 - \mathcal{A}_\mu^1$  is contained in a union of strata of codimension at least  $n(2) = 8$ . Therefore, the inclusion  $G_\mu^1 \rightarrow G_{\mu+\varepsilon}^1$  induces an isomorphism on  $\pi_k$  for  $k \leq 5$  and hence an isomorphism

$$\pi_k(G_\mu^1) \xrightarrow{\cong} \pi_k(\lim G_\mu^1) = \pi_k(\mathcal{D}_0^1)$$

for these  $k$ . The proof of Proposition 1.6 (i) is similar.  $\square$

Part (ii) of these propositions are proved by different methods. Proposition 1.5 (ii) is proved together with Theorem 1.4 (ii) in the next subsection. By contrast Proposition 1.6 (ii) is proved essentially by direct calculation: see §5.2.

### 2.3 Topological changes as $\mu$ passes an integer

We will write  $\mathcal{X}_{\mu,k}$  for the inverse image of  $\mathcal{A}_{\mu,k}$  in  $\mathcal{X}_\mu$ . Thus  $\mathcal{X}_{\mu,k}$  consists of all pairs  $(\omega, J) \in \mathcal{X}_\mu$  for which  $J \in \mathcal{A}_{\mu,k}$ . Note that whenever  $\mathcal{X}_{\mu,k}$  is nonempty the map  $\mathcal{X}_{\mu,k} \rightarrow \mathcal{S}_\mu$  is a fibration whose fiber will be denoted  $\mathcal{J}_{\mu,k}$ . Thus, the contractible space  $\mathcal{J}_\mu$  of almost complex structures tamed by  $\omega_\mu$  is the union of the  $\mathcal{J}_{\mu,k}$ . Since  $\mathcal{J}_\mu$  is open in  $\mathcal{A}(\omega_\mu)$ , each  $\mathcal{J}_{\mu,k} = \mathcal{J}_\mu \cap \mathcal{A}_{\mu,k}$  is a suborbifold of  $\mathcal{J}_\mu$ ,

As  $\mu$  passes the integer  $k$ , a new stratum of codimension  $n = n(k) = 4k - 2 + 2g$  appears in the spaces  $\mathcal{A}_\mu, \mathcal{X}_\mu, \mathcal{J}_\mu$ . Since  $\mathcal{J}_\mu$  is contractible and  $\mathcal{J}_{\mu,k}$  is an orbifold, there is an isomorphism of rational homology

$$H_i(\mathcal{J}_{\mu,k}) \cong H_{i+n-1}(\mathcal{J}_\mu - \mathcal{J}_{\mu,k}).$$

In particular, there is a nonzero element  $\alpha \in H_{n-1}(\mathcal{J}_\mu - \mathcal{J}_{\mu,k})$  that is generated by the sphere linking the new stratum. Since  $\alpha$  is spherical it must represent a nonzero class in the homotopy group  $\pi_{n-1}(\mathcal{J}_\mu - \mathcal{J}_{\mu,k})$ . When  $g = 0$  these elements turn out to generate the whole of the topology change in  $G_\mu$  as  $\mu$  passes  $k$ . It is not clear whether this extends to higher genus. However, as we now see, in some circumstances it is possible to work out the effect of these new elements  $\alpha$  on the rational homotopy of  $G_\mu$ .

**Proposition 2.12** Let  $n = 4k - 2 + 2g$  and suppose that the inclusion  $\mathcal{A}_k^g \rightarrow \mathcal{A}_{k+1}^g - \mathcal{A}_{k+1,k}^g$  induces an isomorphism on  $\pi_i$  for all  $i \leq n$ . Suppose further that the map  $G_\mu^g \rightarrow \mathcal{D}_0^g$  induces an isomorphism on  $\pi_{n-1} \otimes \mathbb{Q}$  for  $\mu = k+1$  and is surjective for  $\mu = k$ . Then there is a nonzero element  $\rho \in \pi_{n-2}(G_k) \otimes \mathbb{Q}$  that vanishes in  $\pi_{n-2}(G_\mu) \otimes \mathbb{Q}$  when  $\mu > k$ .

**Proof:** Consider the following diagram:

$$\begin{array}{ccccccc} \mathcal{J}_{k+1} - \mathcal{J}_{k+1,k} & \rightarrow & \mathcal{X}_{k+1} - \mathcal{X}_{k+1,k} & \rightarrow & \mathcal{S}_{k+1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{J}_{k+1} & \rightarrow & \mathcal{X}_{k+1} & \rightarrow & \mathcal{S}_{k+1}. \end{array}$$

The rows here are fibrations. As already noted,

$$H_{n-1}(\mathcal{J}_{k+1} - \mathcal{J}_{k+1,k}) \cong H_0(\mathcal{J}_{k+1,k}) \cong \mathbb{Q},$$

and is generated by a spherical element  $\alpha \in \pi_{n-1}(\mathcal{J}_{k+1} - \mathcal{J}_{k+1,k})$ . Suppose first that  $\alpha$  goes to 0 in  $\mathcal{X}_{k+1} - \mathcal{X}_{k+1,k}$  and so comes from a nonzero element  $\beta \in \pi_n(\mathcal{S}_{k+1}) \otimes \mathbb{Q}$ , and consider the fibration

$$G_{k+1} \rightarrow \text{Diff}_0(M) \rightarrow \mathcal{S}_{k+1}.$$

The first claim is that  $\beta$  cannot lift to an element  $\beta' \in \pi_n(\text{Diff}_0(M)) \otimes \mathbb{Q}$ . For if it did, because  $\text{Diff}_0(M)$  also acts on  $\mathcal{S}_k$ ,  $\beta$  would be homotopic to an element in

$$\pi_n(\mathcal{S}_k) \otimes \mathbb{Q} = \pi_n(\mathcal{X}_k) \otimes \mathbb{Q} = \pi_n(\mathcal{X}_{k+1} - \mathcal{X}_{k+1,k}) \otimes \mathbb{Q}$$

and so would go to zero under the boundary map from  $\pi_n(\mathcal{S}_{k+1})$  into  $\pi_{n-1}(\mathcal{J}_{k+1} - \mathcal{J}_{k+1,k}) \otimes \mathbb{Q}$ . This contradicts the fact that it has nonzero image  $\alpha$  under this map.

Hence  $\beta$  must give rise to a nonzero element in  $\pi_{n-1}(G_{k+1}) \otimes \mathbb{Q}$  that goes to zero in  $\text{Diff}_0(M)$ . But there are no such elements: by hypothesis, the map  $\pi_{n-1}(G_{k+1}) \otimes \mathbb{Q} \rightarrow \pi_{n-1}(\mathcal{D}_0^g) \otimes \mathbb{Q}$  is an isomorphism, and it follows from the explicit calculation of the groups  $\pi_i(\mathcal{D}_0^g) \otimes \mathbb{Q}$  in §5.1 that  $\pi_*(\mathcal{D}_0^g) \otimes \mathbb{Q}$  injects into  $\pi_*(\text{Diff}_0(M))$ . Hence the original hypothesis must be wrong, i.e.  $\alpha$  must map to a nonzero element  $\alpha'$  in  $\pi_{n-1}(\mathcal{X}_{k+1} - \mathcal{X}_{k+1,k}) \otimes \mathbb{Q} = \pi_{n-1}(\mathcal{X}_k) \otimes \mathbb{Q}$

There are now two possibilities:

- (i) The element  $\alpha' \in \pi_{n-1}(\mathcal{X}_k) \otimes \mathbb{Q}$  comes from  $\gamma \in \pi_{n-1}(\text{Diff}_0(M)) \otimes \mathbb{Q}$ ; or
- (ii)  $\alpha'$  gives rise to a nonzero element  $\alpha'' \in \pi_{n-2}(G_k) \otimes \mathbb{Q}$ .

Note that these elements  $\alpha', \alpha''$  are fragile in the sense that they vanish in  $\mathcal{X}_\mu$  and  $G_\mu$  when  $\mu > k$  since  $\alpha$  does. Thus in case (i) the element  $\gamma \in \pi_{n-1}(\text{Diff}_0(M)) \otimes \mathbb{Q}$  is in the image of  $\pi_{n-1}(G_\mu) \otimes \mathbb{Q}$  for  $\mu > k$  but is not in this image for  $\mu = k$ . Since this contradicts our hypothesis, we must be in case (ii). This completes the proof.  $\square$

### Proof of Theorem 1.4 (ii) and Proposition 1.5 (ii)

Apply the above proposition with  $k = \ell + 1$  and  $g = 0$ , so that  $n = 4k - 2$ . The first hypothesis above is satisfied for all  $k$  since  $\mathcal{A}_k^0 = \mathcal{A}_{k+1}^0 - \mathcal{A}_{k+1,k}^0$  by Corollary 2.9. The second hypothesis holds when  $k \geq 2$  by Theorem 1.4 (i): indeed the lowest homotopy group of  $G_\mu^0$  that changes when  $\mu > k$  increases has dimension  $4k > n - 1$ . The third hypothesis holds for  $k \geq 2$  by Corollary 5.5. This proves Theorem 1.4 (ii).

The proof of Proposition 1.5 (ii) is similar. Take  $k = 1$ . Then  $\mathcal{A}_1^1 = \mathcal{A}_{2,0}^1$  so that the first hypothesis holds. The map  $\pi_3(G_\mu^1) \otimes \mathbb{Q} \rightarrow \pi_3(\mathcal{D}_0^1) \otimes \mathbb{Q}$  is surjective for all  $\mu$  by Proposition 1.6 (ii), and is an isomorphism for  $\mu = 2$  by Proposition 1.5 (i). Note that the results in §5 are independent of the current arguments and so it is permissible to use them here.  $\square$

**Remark 2.13** It was proved in [2] that the elements  $\rho_\ell \in \pi_{4\ell}(G_\mu^0)$  exist only when  $\mu \in (\ell, \ell+1]$ . To prove this in the present context one needs to see that the link  $\alpha \in H_*(\mathcal{J}_\mu - \mathcal{J}_{\mu,\ell})$  cannot be represented in  $\mathcal{J}_\mu - \mathcal{J}_{\mu,\ell} - \mathcal{J}_{\mu,\ell-1}$ , i.e. that its intersection with the stratum  $\mathcal{J}_{\mu,\ell-1}$  is homologically nontrivial. Thus we need information on the homology of the stratum  $\mathcal{J}_{\mu,\ell-1}$ . When the base is a sphere we have this information because the strata are homogeneous spaces of  $G_\mu^0$ . The intersection of  $\alpha$  with  $\mathcal{J}_{\mu,\ell-1}$  is just the link of  $\mathcal{J}_{\mu,\ell}$  in  $\mathcal{J}_{\mu,\ell-1}$ , and the fact that this is homologically essential is a key step in the argument: see [2] Corollary 3.2 and its use in the proof of Proposition 3.7.

### 3 Inflation

This section is devoted to the proofs of Lemma 2.2 and Proposition 2.7.

**Proof of Lemma 2.2.** It remains to prove the claim, i.e. that if  $\sigma$  is any area form on  $\Sigma$  then its pullback  $\pi_J^*(\sigma)$  is  $J$ -semi-tame. To see this, fix a point  $p \in M$  and choose (positively oriented) coordinates  $(x_1, x_2, y_1, y_2)$  near  $p$  so that the fibers near  $p$  have equation  $x_i = \text{const}$ , for  $i = 1, 2$ , and so that the symplectic orthogonal to the fiber  $F_p$  at  $p$

$$Hor_p = \{u : \omega(u, v) = 0, \text{ for all } v \in T_p(F_p)\}$$

is tangent at  $p$  to the surface  $y_1 = y_2 = 0$ . Then, at  $p$ , the form  $\omega_p$  may be written

$$\omega_p = adx_1 \wedge dx_2 + bdy_1 \wedge dy_2$$

for some constants  $a, b > 0$ . Moreover, since  $J_p$  preserves the fibers, we can assume that  $J_p$  acts on  $T_p M$  via the lower triangular matrix

$$J_p = \begin{pmatrix} A & 0 \\ C & J_0 \end{pmatrix}, \quad \text{where} \quad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If  $u \in Hor_p$ , then

$$\omega_p(u, Ju) = adx_1 \wedge dx_2(u, Au) > 0, \quad \text{if } u \neq 0,$$

because  $\omega_p$  tames  $J_p$ . But  $\pi_J^*(\sigma)$  is just a positive multiple of  $dx_1 \wedge dx_2$  at  $p$ . The claim follows.  $\square$

To prove Proposition 2.7, it clearly suffices to prove the following result. (A weaker version, in which the symplectic forms  $\tau_\lambda$  tame a suitable perturbation of  $J$ , is proved in [9].)

**Lemma 3.1 (Inflation)** *Let  $J$  be an  $\tau$ -tame almost complex structure on a symplectic 4-manifold  $(M, \tau_0)$  that admits a  $J$ -holomorphic curve  $Z$  with  $Z \cdot Z \geq 0$ . Then there is a family  $\tau_\lambda$ ,  $\lambda \geq 0$ , of symplectic forms that all tame  $J$  and have cohomology class  $[\tau_\lambda] = [\tau_0] + \lambda a_Z$ , where  $a_Z$  is Poincaré dual to  $[Z]$ .*

We prove this first in the case when  $Z$  has trivial normal bundle.

**Lemma 3.2** Suppose  $\tau_0 = \sigma_Z + dy_1 \wedge dy_2$  in  $Z \times D^2$  where  $(y_1, y_2)$  are rectilinear coordinates on the unit disc  $D^2$ , and let  $J$  be any  $\tau_0$ -tame almost complex structure on  $Z \times D^2$  such that  $Z \times \{0\}$  is  $J$ -holomorphic. Then there is a smooth family of symplectic forms  $\tau_\lambda, \lambda \geq 0$ , on  $Z \times D^2$  such that

(i) each  $\tau_\lambda$  tames  $J$ ;

(ii) for all  $\lambda > 0$ ,

$$\int_{pt \times D} \tau_\lambda = \pi + \lambda;$$

(iii)  $\tau_\lambda = \tau_0$  near  $Z \times \partial D^2$ .

**Proof:** We will construct  $\tau_\lambda$  to have the form

$$\tau_\lambda = \sigma_Z + f_\lambda(r)dy_1 \wedge dy_2,$$

where  $f_\lambda$  is a function of the polar radius  $r = \sqrt{y_1^2 + y_2^2}$  that is chosen to be  $\geq 1$  everywhere and  $= 1$  near  $r = 1$ . The only problem is to make sure that  $\tau_\lambda$  tames  $J$ .

Consider a point  $p = (z, y_1, y_2) \in Z \times D^2$ . We will write elements of  $T_p(Z \times D^2)$  as pairs  $(u, v)$  where  $u$  is tangent to  $Z$  and  $v$  is tangent to  $D^2$ , and will choose these coordinates so that

$$\tau_\lambda((u, v), (u', v')) = u^T J_0^T u' + f_\lambda v^T J_0^T v',$$

where  ${}^T$  denotes transpose and  $J_0$  is as before. With respect to the obvious product coordinates, we can write  $J_p$  in block diagonal form

$$J_p = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where  $A, B, C, D$  are  $2 \times 2$  matrices. Therefore, we have to choose  $f = f_\lambda(r)$  so that

$$\tau_\lambda((u, v), J_p(u, v)) = u^T J_0^T A u + u^T J_0^T B v + f v^T J_0^T C u + f v^T J_0^T D v \geq 0$$

for all  $(u, v) \in T_p(Z \times D^2)$ . Because  $B = C = 0$  when  $r = 0$  and  $\tau_0$  tames  $J$ , there are constants  $c, c'$  so that the following estimates hold in some neighborhood  $\{r \leq r_0\}$  of  $Z \times \{0\}$ :

$$\begin{aligned} \|u\|^2 &\leq c u^T J_0^T A u, & \|v\|^2 &\leq c v^T J_0^T D v \\ \|u^T J_0^T B v\| &\leq c' r (\|u^2\| + \|v\|^2) & \|v^T J_0^T C u\| &\leq c' r (\|u^2\| + \|v\|^2). \end{aligned}$$

Hence, because  $f \geq 1$  always and  $f v^T = \sqrt{f}(\sqrt{f} v^T)$ ,

$$\begin{aligned} \|u^T J_0^T B v\| + \|f v^T J_0^T C u\| &\leq c' r (\|u^2\| + \|v\|^2) + \sqrt{f} c' r (\|u^2\| + f \|v\|^2) \\ &\leq c' r (1 + \sqrt{f}) (\|u^2\| + f \|v\|^2) \\ &\leq c c' r (1 + \sqrt{f}) (u^T J_0^T A u + f v^T J_0^T D v). \end{aligned}$$

Therefore,  $\tau_\lambda$  will tame  $J_p$  provided that  $cc'r(1 + \sqrt{f}) < 1$ . Hence it suffices that

$$f = f_\lambda(r) \leq \frac{1}{\alpha r^2}, \quad \text{where } \alpha = (2cc')^2$$

when  $r \leq r_0$ , and  $f = 1$  for  $r \geq r_0$ . But for functions of this type, the integral

$$\int_{D^2} f_\lambda dx \wedge dy$$

can be arbitrarily large. The result follows.  $\square$

The proof in the general case is similar, but it uses a more complicated normal form for  $\tau_0$  near  $Z$ . Think of a neighborhood  $\mathcal{N}(Z)$  of  $Z$  as the unit disc bundle of some complex line bundle over  $Z$ , let  $r$  be the radial coordinate in this line bundle and choose a connection form  $\alpha$  for the associated circle bundle so that  $d\alpha = -\kappa\pi^*(\sigma_Z)$ . (Here  $\kappa = Z \cdot Z > 0$  if we suppose  $\sigma_Z$  normalized to have integral 1 over  $Z$ .) Then, by the symplectic neighborhood theorem, we may identify  $\tau_0$  with the form

$$\pi^*(\sigma_Z) + d(r^2\alpha) = (1 - \kappa r^2)\pi^*\sigma_Z + 2rdr \wedge \alpha$$

in some neighborhood  $r \leq r_0$ .

We will take  $\tau_\lambda$  to have the form

$$\pi^*(\sigma_Z) + d(r^2\alpha) - d(f_\lambda(r)\alpha) = (1 - \kappa r^2 + \kappa f_\lambda)\pi^*\sigma_Z + (2r - f'_\lambda)dr \wedge \alpha$$

for a suitable nonincreasing function  $f_\lambda$  with support in  $r \leq r_0$ . Thus  $f_\lambda(0) > 0$  and we assume that  $f_\lambda$  is constant very near 0. The form  $-d(f_\lambda(r)\alpha)$  represents the positive multiple  $f_\lambda(0)/\kappa$  of  $a_Z$ . Hence we need to see that we can choose  $f_\lambda$  so that  $f_\lambda(0)$  is arbitrarily large and so that  $\tau_\lambda$  tames  $J$ .

At each point  $p$ , split  $T_p\mathcal{N}(Z)$  into a direct sum  $E_H \oplus E_F$  where the horizontal space  $E_H$  is in the kernel of both  $dr$  and  $\alpha$  and where  $E_F$  is tangent to the fiber. Then these subspaces are orthogonal with respect to  $\tau_\lambda$ , for all  $\lambda$  and we may choose bases in these spaces so that for  $(u, v) \in E_H \oplus E_F$ ,

$$\tau_0((u, v), (u', v')) = u^T J_0^T u' + v^T J_0^T v'$$

as before. Then  $\tau_\lambda((u, v), (u', v')) = au^T J_0^T u' + bv^T J_0^T v'$ , where

$$a = 1 + \frac{\kappa f_\lambda}{1 - \kappa r^2} \geq 1, \quad b = 1 - \frac{f'_\lambda}{2r} \geq 1.$$

Arguing as above, we find that  $\tau_\lambda$  will tame  $J_p$  provided that  $cr(\sqrt{a} + \sqrt{b}) < 1$ , where  $c$  is a constant depending only on  $J$ . Such an inequality is satisfied if  $-f'_\lambda \leq c'/r$  for a suitable constant  $c'$  since then  $f_\lambda = \text{const} - c'\log(r)$ . It follows that we can choose  $f_\lambda$  so that  $J$  is tamed and  $f_\lambda(0)$  is arbitrarily large. Further details are left to the reader.  $\square$

**Corollary 3.3** *If  $J \in \mathcal{A}_\mu$  admits a curve  $Z$  in class  $pA + qF$ , where  $p > 0, q \geq 0$ , then  $J \in \mathcal{A}_{\mu'}$  for every  $\mu' > q/p$ .*

**Proof:** Choose a form  $\tau_0 \in \mathcal{S}_\mu$  that tames  $J$  and let

$$\sigma_\lambda = \frac{1}{1 + \lambda p} \tau_\lambda,$$

where  $\tau_\lambda$  is the family of forms taming  $J$  constructed as above. Then

$$\sigma_\lambda(F) = 1, \quad \sigma_\lambda(A) = \frac{\mu + \lambda q}{1 + \lambda p} = \frac{q}{p} + \varepsilon_\lambda$$

for arbitrarily small  $\varepsilon_\lambda > 0$ . The result is now immediate.  $\square$

## 4 Finding embedded holomorphic curves

Our aim in this section is to prove Lemma 2.8 and 2.10. To do this we have to show that appropriate homology classes  $B$  have embedded  $J$ -holomorphic representatives for *each* given  $J$ . Here is the basic method that we will use.

Given a homology class  $B$ , let  $\text{Gr}(B)$  denote the Gromov invariant of  $B$  as defined by Taubes. This counts the number<sup>4</sup> of embedded  $J$ -holomorphic curves in class  $B$  through  $k(B)$  generic points for a fixed generic  $J$ , where

$$k(B) = \frac{1}{2}(c_1(B) + B^2).$$

Therefore, whenever  $\text{Gr}(B) \neq 0$  we know that there have to be embedded  $J$ -holomorphic curves in class  $B$  for generic  $J$ . Hence, by Gromov compactness, there is for each  $J$  *some*  $J$ -holomorphic cusp-curve (or stable map)<sup>5</sup> that represents  $B$  and goes through an arbitrary set of  $k(B)$  points, and our task is to show that at least one of these representatives is an embedded curve. We do this by showing that there are not enough degenerate representatives to go through  $k(B)$  generic points.

One very important point is that, by the work of Hofer–Lizan–Sikorav [5], embedded curves in class  $B$  are regular<sup>6</sup> whenever  $c_1(B) > 0$ . We shall need the analog of this result for non-embedded curves, which has been worked out by Ivashkovich–Shevchishin [6]. The relevant condition is  $c_1(B) > g(B)$ , where  $g(B)$  is the genus of an embedded representative of  $B$ .

This method works only because ruled surfaces have many nonzero Gromov invariants. Indeed, it was proved by Li–Liu [10] that, if  $M = \Sigma \times S^2$  where  $g > 0$  and  $B = pA + qF$ , then

$$\text{Gr}(B) = (p+1)^g, \quad \text{provided that } k(B) \geq 0.$$

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<sup>4</sup>This statement holds provided that the class  $B$  has no representatives by multiply covered tori of zero self-intersection. It is easy to check that this hypothesis holds in all cases considered here. Observe also that each curve is equipped with a sign, and that one takes the algebraic sum.

<sup>5</sup>Here we use the word “curve” to denote the image of a connected smooth Riemann surface under a  $J$ -holomorphic map, and reserve the word “cusp-curve” or stable map for a connected union of more than one  $J$ -holomorphic curve.

<sup>6</sup>A  $J$ -curve  $u : \Sigma \rightarrow M$  is said to be regular if the linearization  $Du$  of the generalized Cauchy–Riemann operator at  $u$  is surjective. If this is the case for all  $J$ -curves in some moduli space, then this moduli space is a manifold of the “correct” dimension, i.e. its (real) dimension equals the index  $2k(B)$  of  $Du$ . Thus the moduli space of  $B$ -curves with  $k(B)$  marked points has real dimension  $4k(B)$ , the same as the dimension of  $M^{k(B)}$ . To a first approximation, the Gromov invariant is just the degree of the evaluation map from this pointed moduli space to  $M^{k(B)}$ . See [15, 13] for further information.

In particular,  $\text{Gr}(B) \neq 0$  provided that  $q \geq g - 1$ . When  $g = 0$ ,  $\text{Gr}(B) = 1$  for all classes  $B$  with  $p, q \geq 0$  and  $p + q > 0$ .

The following result is well known. We sketch the proof because most references consider only the case of generic  $J$ .

**Lemma 4.1** *For each  $J \in \mathcal{A}_\mu$ ,  $M = \Sigma \times S^2$  admits a smooth foliation by  $J$ -holomorphic spheres in class  $F$ .*

**Proof:** The above remarks imply that  $\text{Gr}(F) \neq 0$  for all  $g$ . Hence the fiber class always has some  $J$ -holomorphic representative. When  $g > 0$  any such representative has to be a curve, rather than a cusp-curve, since  $F$  is a generator of the spherical part of  $H_2(M)$ . Hence the moduli space of  $J$ -holomorphic curves in class  $F$  is compact and has real dimension  $2k(F) = 2$ . Since  $F \cdot F = 0$ , the curves have to be disjoint by positivity of intersections, and they are embedded by the adjunction formula. The proof that they form the leaves of a smooth foliation is a little more subtle and may be found in [8] Prop. 4.12. See also [5].

When  $g = 0$  we must rule out the possibility that  $F$  can be represented by a cusp-curve. If so,  $F$  would decompose as a sum  $B_1 + \dots + B_\ell$ , where each  $B_i = p_i A + q_i F$  has a spherical  $J$ -holomorphic representative and  $\ell > 1$ . Thus  $\omega_\mu(B_i) = \mu p_i + q_i > 0$  for all  $i$ . Since also  $\sum_i \omega(B_i) = 1$  and  $\ell > 1$ ,  $\omega(B_i) < 1$  for all  $i$ . Therefore, since  $\mu \geq 1$ , for each  $i$   $p_i$  and  $q_i$  must both be nonzero and have opposite signs. However  $B_i \cdot B_j < 0$  if  $p_i, p_j$  have the same sign. Therefore, by positivity of intersections,  $\ell = 2$  and  $p_1, p_2$  have opposite signs. Further the classes  $B_1, B_2$  and hence  $F$  have unique representatives, contradicting the fact that there is an  $F$  cusp-curve through every point.  $\square$

### Proof of Lemma 2.8

We have to show that when  $g = 0$  and  $J \in \mathcal{A}_{\mu,\ell}$  the class  $B = A + \ell F$  has a  $J$ -holomorphic representative. By the above,  $\text{Gr}(B) = 1$  and  $k(B) = 1 + 2\ell$ . Hence there is at least one  $J$ -holomorphic representative of the class  $B$  through each set of  $1 + 2\ell$  points. Because  $A - \ell F$  is represented, it follows from positivity of intersections that no class of the form  $A + qB$  with  $q < \ell, q \neq -\ell$  can be represented. Therefore, the only cusp-curves that represent  $B$  must be the union of the fixed negative curve  $C_J$  in class  $A - \ell F$  with  $2\ell J$ -fibers. But if we choose the  $1 + 2\ell$  points so that none lies on  $C_\ell$  and no two lie in the same  $J$ -fiber then there is no such cusp-curve through these points. Hence there has to be an embedded curve through these points.  $\square$

If one tries the same argument when  $g > 0$  we find that  $c_1(B) > 0$  when  $\ell > g - 1$  and that  $k(B) = 1 + 2\ell - g$ . Hence there are enough cusp-curves of the form  $C_\ell$  plus  $2\ell$  fibers to go through  $k(B)$  generic points. However, now  $\text{Gr}(B) = g + 1$  and one might be able to show that these cusp-curves cannot account for all the needed elements. For example, when  $g = \ell = 1$ , there is only one cusp-curve through  $k(B) = 2$  generic points, and so if one could show this always has “multiplicity 1”, there would have to be another embedded representative of the class  $B = A + F$ . To carry such an argument through would require quite a bit of analysis. Therefore, for now, we will only prove the following partial results.

**Lemma 4.2** *(i) Every  $J \in \mathcal{A}_{\mu,1}^1$  admits an embedded representative in either  $A + F$  or  $2A + 3F$ .  
(ii) Every  $J \in \mathcal{A}_{\mu,0}$  admits an embedded representative in the class  $B = A + qF$  for some integer  $q \leq g/2$ .*

**Proof:** We first prove (i), so  $g = 1$ . We shall be considering classes of the form  $A + qF$  and  $2A + qF$ . Since  $(A + qF) \cdot F = 1$ , it follows from positivity of intersections that any  $J$ -holomorphic representative  $C$  of this class  $A + qF$  must either intersect a  $J$ -holomorphic fiber transversally in a single point or must contain this fiber completely. Thus each component of  $C$  must be embedded. If  $C$  represents the class  $2A + qF$ , then  $C$  could have double points or critical points of order 1, i.e. places where the derivative  $du$  of the parametrizing map  $u$  has a simple zero. (The critical points cannot have higher order because  $C \cdot F = 2$ .) Since each such critical point of  $C$  reduces its genus by 1 there can be at most  $g(2A + qF)$  such points, where  $g(B)$  denotes the genus of an embedded representative of  $B$ . Thus it follows from Theorem 2 of [6] that the curve  $C$  will belong to a smooth moduli space of the “correct” dimension provided that  $c_1(2A + qF) > g(2A + qF)$ .

Now, when  $g = 1$ ,

$$\begin{aligned} k(A + qF) &= 2q, & c_1(A + qF) &= 2q, & g(A + qF) &= 1, \\ k(2A + qF) &= 3q, & c_1(2A + qF) &= 2q, & g(2A + qF) &= 1 + q. \end{aligned}$$

Therefore  $c_1(2A + qF) > g(2A + qF)$  provided that  $q > 1$ . In this case  $k(2A + qF) > 0$  so that  $\text{Gr}(2A + qF) > 0$ .

Because the classes  $F$  and  $A - F$  have  $J$ -representatives by hypothesis, the only other classes  $pA + qF$  that can be represented have  $p \geq 0$  and  $q \geq p$ . Thus, the possible representatives for the class  $2A + 3F$  have one of the following types.

- (a) an embedded representative of  $2A + 3F$ ;
- (b) a representative of  $2A + 3F$  with some critical or double points;
- (c) the union of two embedded curves in class  $A + q_iF$ , where  $q_i \geq 1$ , with  $3 - q_1 - q_2$  fibers;
- (d) the union of the  $A - F$  curve, together with an embedded  $A + qF$  curve, with  $q \geq 1$ , and  $4 - q$  fibers;
- (e) a 2-fold copy of the  $A - F$  curve together with 5 fibers.

We claim that if (a) does not hold, there is an embedded curve in class  $A + F$ . Suppose not. Then there can be no cusp-curves of type (c) since one of the  $q_i$  would have to be 1. Therefore it remains to show that there are not enough cusp-curves of types (b), (d) or (e) to go through  $k(2A + 3F) = 9$  generic points.

Let us consider these types in turn. Since  $c_1(B) > g(B)$  when  $B = 2A + mF$  any curve of type (b) is regular and so belongs to a moduli space of (complex) dimension strictly smaller than 9: indeed, it is easy to see from [6] that the difference is precisely the number of critical points.

Now consider cusp-curves of type (d). In this case, the  $A + qF$  curve would be regular since  $q > 1$ . Hence this cusp-curve could go through at most

$$k(A + qF) + 4 - q \leq k(A + 4F) = 8$$

points. Finally, curves of type (e) go through at most 5 generic points. This proves (i).

The proof of (ii) is easier. Let  $m = k$  where  $g = 2k + 1$  or  $2k$ . Then  $k(A + mF) \geq 0$ . Therefore the class  $A + mF$  has to have some  $J$ -holomorphic representative. If  $J \in \mathcal{A}_{\mu,0}$ , this must be the union of an embedded curve in class  $A + kF$  with  $m - k$  fibers for some  $k \leq m$  since there are no negative  $J$  curves.  $\square$

**Corollary 4.3** *Lemma 2.10 holds.*

**Proof:** Apply Proposition 2.7. □

**Remark 4.4** The argument in Lemma 4.2 can be generalized to the case  $g > 1$ . The results are not very sharp because the requirement that  $c_1(B) > g(B)$ , which is used to ensure that the curve is embedded, forces us to consider only classes  $B = pA + qF$  with  $p \leq 2$ . However, it is quite possible that we might be able to find an embedded  $J$ -curve under weaker hypotheses. For example, if  $\ell \geq g$ , any  $J$ -curve in class  $B$  would have to have  $q/p \geq \ell > g - 1$  which is enough to imply that  $c_1(B) > 0$ . Hence all the  $J$ -curves in class  $B$  would be regular. It is then not hard to adapt the above argument to show that there is a nonempty moduli space of curves in some such class  $B$  with  $q/p$  arbitrarily close to  $\ell$ . It is possible that one could show that a generic element of this moduli space must be embedded. If so, then we would be able to prove the analog of part (i) of Theorem 1.4 when  $\ell > g$ .

## 5 The groups $G_\mu^g$

We first study the topology of the groups  $\mathcal{D}^g$  and  $\mathcal{D}_0^g$ , and then prove part (ii) of Proposition 1.6.

### 5.1 The groups $\mathcal{D}^g$ and $\mathcal{D}_0^g$ .

Consider the group  $\mathcal{D}^g$  of fiber preserving diffeomorphisms of  $\Sigma \times S^2$ , and let  $\iota : \mathcal{D}^g \rightarrow \text{Diff}(\Sigma \times S^2)$  be the inclusion.

**Lemma 5.1** *The map  $\iota_* : \pi_0(\mathcal{D}^g) \rightarrow \pi_0(\text{Diff}(\Sigma \times S^2))$  is injective.*

**Proof:** Clearly,

$$\mathcal{D}^g \simeq \text{Diff}(\Sigma) \times \text{Map}(\Sigma, \text{SO}(3)).$$

Since  $\pi_2(\text{SO}(3)) = 0$ , any map  $\Sigma \rightarrow \text{SO}(3)$  that induces the zero map on  $\pi_1$  is homotopic to a constant map. Hence

$$\pi_0(\mathcal{D}^g) = \pi_0(\text{Diff}(\Sigma)) \times \text{Hom}(\pi_1(\Sigma), \text{SO}(3)) = \pi_0(\text{Diff}(\Sigma)) \times (\mathbb{Z}/2\mathbb{Z})^{2g}.$$

Next observe that there are maps

$$\text{Diff}(\Sigma \times S^2) \rightarrow \text{Map}(\Sigma \times S^2, S^2) = \text{Map}(\Sigma, \text{Map}_1(S^2, S^2)) \rightarrow \text{Map}(\Sigma, \text{SO}(3)),$$

where the first map takes a diffeomorphism  $\phi$  to its composite with the projection onto  $S^2$ , and the second exists because there is a projection of the space  $\text{Map}_1(S^2, S^2)$  of degree 1 self-maps of the sphere onto  $\text{SO}(3)$  (see [1].) It follows that the part of  $\pi_0(\text{Diff}(\Sigma))$  coming from  $\text{Map}(\Sigma, \text{SO}(3))$  injects into  $\pi_0(\text{Diff}(\Sigma \times S^2))$ . The rest of  $\pi_0(\text{Diff}(\Sigma))$  can be detected by its action on the set of conjugacy classes in the group  $\pi_1(\Sigma \times S^2)$ , and hence also injects. □

The following corollary was used in §2.1.

**Corollary 5.2**  $\mathcal{D}^g \cap \text{Diff}_0(\Sigma \times S^2) = \mathcal{D}_0^g$ .

Now consider the identity component  $\mathcal{D}_0^g$ . Clearly

$$\begin{aligned}\mathcal{D}_0^0 &\simeq \text{SO}(3) \times \text{Map}_0(T^2, \text{SO}(3)), & \text{when } \Sigma &= S^2, \\ \mathcal{D}_0^1 &\simeq T^2 \times \text{Map}_0(T^2, \text{SO}(3)), & \text{when } \Sigma &= T^2, \\ \mathcal{D}_0^g &\simeq \text{Map}_0(\Sigma, \text{SO}(3)), & \text{otherwise,}\end{aligned}$$

where  $\text{Map}_0$  denotes the component containing the constant maps. Consider the evaluation map  $\text{Map}_0(\Sigma, \text{SO}(3)) \rightarrow \text{SO}(3)$ . Because the elements of  $\text{SO}(3)$  lift to the constant maps,  $\text{Map}_0$  is a product:

$$\text{Map}_0(\Sigma, \text{SO}(3)) \cong \text{SO}(3) \times \text{Map}_*(\Sigma, \text{SO}(3)),$$

where  $\text{Map}_*$  is the connected space of all based null-homotopic maps. Moreover, since every such map  $\Sigma \rightarrow \text{SO}(3)$  has a unique lift to  $S^3$ ,

$$\text{Map}_*(\Sigma, \text{SO}(3)) = \text{Map}_*(\Sigma, S^3).$$

We write  $\Omega(X) = \text{Map}_*(S^1, X)$  for the identity component of the based loop space of  $X$ , and  $\Omega^2(X) = \text{Map}_*(S^2, X)$  for the identity component of the double loop space.

**Lemma 5.3**  $\text{Map}_*(\Sigma, S^3)$  is homotopy equivalent to the product of  $\Omega^2(S^3)$  with  $2g$  copies of  $\Omega(S^3)$ .

**Proof:** Choose loops  $\gamma_1, \dots, \gamma_{2g}$  in  $\Sigma$  that represent a standard basis for  $H_1(\Sigma)$  and choose corresponding projections  $pr_j : \Sigma \rightarrow S^1$  such that  $pr_j$  is injective on  $\gamma_j$  and maps each  $\gamma_i, i \neq j$ , to the base point  $id \in S^3$ . Each element  $h \in \text{Map}_{0,*}(\Sigma, S^3)$  determines an element  $(f_i) = (h|_{\gamma_i} \circ pr_j)$  of the  $2g$ -fold product  $(\Omega(S^3))^{2g}$  by restriction to the loops  $\gamma_i$ . Then, if  $\cdot$  denotes the product of maps coming from the group structure on  $S^3$ , the map

$$h' = h \cdot f_1^{-1} \cdot \dots \cdot f_{2g}^{-1}$$

takes each loop  $\gamma_i$  to the base point in  $S^3$  and so can be considered as an element of  $\Omega^2(S^3)$ . It is not hard to check that the map  $h \mapsto (f_i, h')$  is a homotopy equivalence.  $\square$

**Corollary 5.4** The vector space  $\pi_i(\mathcal{D}_0^g) \otimes \mathbb{Q}$  has dimension 1 when  $i = 0, 1, 3$  except in the cases  $i = g = 1$  when the dimension is 3 and  $g = 0, i = 3$  when the dimension is 2. It has dimension  $2g$  when  $i = 2$ , and is zero otherwise.

**Corollary 5.5** When  $\mu > 1$  the map  $G_\mu^0 \rightarrow \mathcal{D}_0^0$  induces a surjection on rational homotopy.

**Proof:** This follows immediately from Proposition 1.3, since the elements in  $\pi_3(\mathcal{D}_0^g) \otimes \mathbb{Q}$  lift to  $G_\mu^0$  for all  $\mu$ .  $\square$

**Corollary 5.6** For all  $g > 0$  the map  $G_\mu^g \rightarrow \mathcal{D}_0^g$  induces a surjection on  $\pi_2 \otimes \mathbb{Q}$  for all  $\mu > 0$ .

**Proof:** We just have to exhibit explicit representatives in  $G_\mu^g$  of the elements in  $\pi_2$  for any  $\mu > 0$ . For definiteness, consider the case  $g = 1$ . Then the generator of  $\pi_2(\mathcal{D}_0^1)$  corresponding to the projection of  $T^2$  onto its first factor is

$$S^2 \times T^2 \rightarrow \text{SO}(3) : (z, s, t) \mapsto R_{z,s}$$

where  $R_{z,s}$  is the rotation of  $S^2$  by angle  $s$  about the axis through  $z$ . Consider the corresponding family of diffeomorphisms

$$\rho_z : T^2 \times S^2 \rightarrow T^2 \times S^2 : (s, t, w) \mapsto (s, t, R_{z,s}w), \quad z \in S^2.$$

Then  $\rho_z^*(\omega_\mu) = \omega_\mu + ds \wedge \alpha_s$  where  $\alpha_s$  is an exact 1-form on  $S^2$  for each  $s$ . Thus  $\rho_z^*(\omega_\mu)$  can be isotoped to  $\omega_\mu$  via the symplectic forms  $\omega_\mu + \nu ds \wedge \alpha_s$ , where  $\nu \in [0, 1]$ . It follows, using Moser's argument, that the family  $\rho_z$  can be homotoped into  $G_\mu^1$  for any  $\mu$ .  $\square$

**Remark 5.7** The whole analysis of the groups  $G_\mu^0$  is based on Gromov's result that when the two spheres have equal size (i.e. when  $\mu = 1$ )  $G_1^0$  is homotopy equivalent to the stabilizer  $\text{Aut}(J_{\text{split}}) = \text{SO}(3) \times \text{SO}(3)$  of the product almost complex structure. More precisely he showed that the (contractible) space  $\mathcal{J}_1$  is homotopy equivalent to the coset space  $G_1^0/\text{Aut}(J_{\text{split}})$ . It is shown in [2] that this continues to hold for the other strata  $\mathcal{J}_{\mu,\ell}$  in  $\mathcal{J}_\mu$ , i.e. each stratum  $\mathcal{J}_{\mu,\ell}$  is homotopy equivalent to some homogeneous space  $G_\mu^0/K_\ell$ , where the Lie group  $K_\ell$  is the stabilizer of an integrable element in  $\mathcal{J}_{\mu,\ell}$ . The basic reason for this is that the topological type of the space of all  $J$ -holomorphic curves does not change as  $J$  varies in a stratum  $\mathcal{J}_{\mu,\ell}$ .

It is conceivable that when  $g > 0$  a similar result continues to hold for some of the  $\mathcal{J}_{\mu,\ell}$ , for example if  $\mu$  and  $\ell$  are sufficiently large. However, it does not hold when  $\ell = 0$ ,  $\mu \leq 1$ . In this case, because  $\mathcal{J}_{\mu,0} = \mathcal{J}_\mu$  is contractible, such a statement would imply that  $G_\mu$  has the homotopy type of the stabilizer of some almost complex structure  $J$  and so would be homotopy equivalent to a Lie group. This is not true because  $\pi_2(G_\mu) \otimes \mathbb{Q} \neq 0$ . Note also that in this case the topological type of the space of  $J$ -holomorphic curves may change as  $J$  varies in  $\mathcal{J}_{\mu,0}$ . For example, when  $g = 1$ , although the Gromov invariant of the class  $A = [T^2 \times pt]$  is 2, there can be arbitrarily many pairs of  $J$ -holomorphic tori in class  $A$  that cancel out in the algebraic sum: see the discussion at the end of [M3]§1.

## 5.2 Relations between $G_\mu$ and $\mathcal{D}_0^g$

It remains to prove part (ii) of Proposition 1.6 that claims that the map  $G_\mu^g \rightarrow \mathcal{D}_0^g$  induces a surjection on rational homotopy when  $g > 0, \mu > 0$ . It follows easily from the results in §5.1 (see particularly Corollary 5.6) that this holds in all dimensions except possibly for 1. Hence it suffices to show:

**Proposition 5.8** *The map  $\pi_1(G_\mu^g) \otimes \mathbb{Q} \rightarrow \pi_1(\mathcal{D}_0^g) \otimes \mathbb{Q}$  is surjective for all  $g > 0, \mu > 0$ .*

**Proof:** Recall that  $\pi_1(\mathcal{D}_0^g) \otimes \mathbb{Q}$  has dimension 1 except when  $g = 1$  when there are two extra dimensions coming from translations of the base. Since these translations lie in  $G_\mu^1$  for all  $\mu$ , we only need to show that the generator of  $\pi_1(\mathcal{D}_0^g)$  coming from  $\pi_1(\Omega^2(S^3))$  lifts to the  $G_\mu^g$ .

As mentioned in §1.1 when  $g = 0$  this generator can be explicitly represented by the circle action

$$\phi_t : S^2 \times S^2 \rightarrow S^2 \times S^2 : (z, w) \mapsto (z, R_{z,t}(w)), \quad t \in [0, 1],$$

where  $R_{z,t}$  is the rotation of the fiber sphere  $S^2$  through the angle  $2\pi t$  about the axis through the point  $z \in S^2$ . Hence the corresponding generator of  $\pi_1(\mathcal{D}_0^g)$  can be represented by the circle

$$\phi_t^g : \Sigma \times S^2 \rightarrow \Sigma \times S^2 : (z, w) \mapsto (z, R_{\rho(z),t}(w)), \quad t \in [0, 1]$$

where  $\rho : \Sigma \rightarrow S^2$  has degree 1. Therefore, we just have to see that  $\{\phi_t^g\}$  can be homotoped into  $G_\mu^g$  for all  $g > 0, \mu > 0$ , i.e. that the orbits of its action on  $\mathcal{S}_\mu^g$  are contractible. Equivalently, we need to see that its orbits contract in  $\mathcal{A}_\mu^g$  for all  $g > 0, \mu > 0$ .

In fact, we shall see that the circle action on  $\mathcal{A}_\mu^g$  has a fixed point  $J_0^g \in \mathcal{A}_{\mu,1}^g$  when  $\mu > 1$ , so that its orbits obviously contract in this case. If we choose  $J \in \mathcal{A}_{\mu,0}^g$  close to  $J_0^g$  its orbit lies in some normal slice  $D$  to the stratum  $\mathcal{A}_{\mu,1}^g$  at  $J_0^g$ . Since  $D$  has dimension at least 4 when  $g > 0$ , the orbits are contractible in  $\mathcal{A}_{\mu,0}^g$  also. This completes the proof when  $g = 1$ , since Lemma 2.10 says that the sets  $\mathcal{A}_{\mu,0}^1$  are the same for all  $\mu > 0$ . However, the latter statement may not hold in higher genus and so the proof requires more work. The idea is to pull a contracting homotopy back from  $\mathcal{A}_{\mu,0}^1$  to  $\mathcal{A}_{\mu,0}^g$ .

If  $g' > g$  an arbitrary almost complex structure  $J$  on  $\Sigma_g \times S^2$  does not of course lift to  $\Sigma_{g'} \times S^2$ . However, suppose that  $J$  splits as a product on some open neighborhood  $U \times S^2$  of some fixed fiber  $F_* = * \times S^2$  and choose a smooth map  $\sigma : \Sigma_{g'} \rightarrow \Sigma_g$  that is bijective except over the set  $\sigma^{-1}(*)$ . Then, if  $\bar{V}$  is a small closed disc centered on  $*$  that is contained in  $U$ , there is an almost complex structure  $J'$  on  $\Sigma_{g'} \times S^2$  that equals the pullback of  $J$  via  $\sigma \times id$  on  $\sigma^{-1}(T^2 - \bar{V})$  and is a product on  $\sigma^{-1}(U) \times S^2$ . Note that  $J'$  is tamed by a symplectic form of the type

$$\omega' = (\sigma \times id)^*(\omega) + pr^*(\tau)$$

where  $\tau$  is a nonnegative 2-form on  $\Sigma_{g'}$  with support near  $\sigma^{-1}(*)$  and  $pr$  is the projection. Thus  $J' \in \mathcal{A}_{\mu'}^{g'}$  for some  $\mu' > \mu$ . The second condition below is introduced to give control over  $\mu'$ .

We will say that  $J \in \mathcal{A}_{\mu,0}^g$  is *normalized* if it satisfies the following conditions:

- (i) it splits as a product near  $F_*$ ,
- (ii) there is a  $J$ -holomorphic curve  $Z$  in class  $A$  that is flat near  $F_*$ , i.e. it coincides with some flat section  $\Sigma_g \times pt$  near  $F_*$ .

As explained above, the first condition allows  $J$  to be pulled back to an element  $J'$  of  $\mathcal{A}_{\mu'}^{g'}$  for some  $\mu' > \mu, g' > g$ . Moreover,  $J'$  is necessarily in  $\mathcal{A}_{\mu',0}^{g'}$ , since it clearly admits a curve  $Z'$  (the pullback of  $Z$ ) in class  $A$ . Inflating along  $Z'$  we find that  $J' \in \mathcal{A}_{\varepsilon,0}^{g'}$  for all  $\varepsilon > 0$ .

We construct below, for some  $\mu > 1$ , a 2-dimensional family  $J_w, w \in D$ , of elements in  $\mathcal{A}_\mu^0$  that intersect the stratum  $\mathcal{A}_{\mu,1}^0$  transversally at  $w = 0$  and are normalized for  $w \neq 0$  via the curves  $C_w$ . This normalization is uniform in the sense that both conditions (i) and (ii) are satisfied over some fixed neighborhood of the fiber  $F_*$ . Thus the central element  $J_0$  also satisfies condition (i) above, and as  $w \rightarrow 0$  the curves  $C_w$  converge to the union of the

$J_0$ -holomorphic curve  $C_0$  in class  $A - F$  with a fixed fiber  $F_0$  distinct from  $F_*$  where  $C_0$  is flat near  $F_*$ . Moreover,  $\phi_t^0$  acts on the  $J_w$  via

$$(\phi_t^0)_*(J_w) = J_{e^{it}w}.$$

This construction permits us to complete the proof as follows. We can define the pullbacks  $(C_w^1, J_w^1)$  of the pairs  $(C_w, J_w)$  to  $T^2 \times S^2$  so that the  $J_w^1$  all lie in some set  $\mathcal{A}_{\mu'}^1$  and the circle acts as above. By construction the 2-disc  $\{J_w^1 : w \in D\}$  is normal to the stratum  $\mathcal{A}_{\mu,1}^1$ , and so sits inside a transverse slice of dimension 4 consisting of elements  $J_v^1$  where  $v = (w, w') \in D'' = D \times D'$ . By the remark at the end of the proof of Lemma 2.6 we can assume that  $J_{(w,w')}^1 = J_{w,0}^1$  near  $F_*$ , so that the  $J_v^1$  satisfy condition (i) above. Moreover, assuming as we may that  $J_0$  is a generic complex structure on the torus  $C_0$ , we can extend the family of  $J_w^1$ -holomorphic curves  $C_w^1$  to all small  $w \in D''$  by the process of gluing as in [14].<sup>7</sup> Here we are making the curves  $C_v^1$  by gluing  $C_0$  to the fiber  $F_0$ . Therefore, because the  $C_v^1$  converge  $C^\infty$  to  $C_0 \cup F_0$  away from the point  $C_0 \cap F_0$ , they will be approximately flat near  $F_*$  and will become flatter as  $|w'| \rightarrow 0$ .

Now suppose that we want to show that the orbits of  $\{\phi_t^g\}$  contract in  $\mathcal{A}_\mu^g$  for some particular  $g$  and  $\mu > 0$ . We can pull the family  $(C_v^1, J_v^1)$  back to a family  $(C_v^g, J_v^g)$  where  $J_v^g \in \mathcal{A}_{\mu'}$  for all  $v \in D$ . The curve  $C_v^g$  will now be only approximately  $J_v^g$ -holomorphic, but its deviation from being holomorphic will tend to 0 as  $|w'| \rightarrow 0$ . This means that inflating along  $C_v^g$  will give rise to symplectic forms  $\omega_\lambda \in \mathcal{S}_\lambda$  that tame  $J_v^g$  for some restricted interval  $\lambda_0 \leq \lambda \leq \mu'$ , where we can make  $\lambda_0$  as close to 0 as we want by making  $|w'|$  sufficiently close to 0. Define

$$W_\delta = \{v \in D'' - \{0\} : |w'| \leq \delta\}.$$

The above discussion shows that for any  $\mu > 0$  there is  $\delta > 0$  so that the pullbacks  $J_v^g$  of the elements  $J_v^1, v \in W_\delta$  lie in  $\mathcal{A}_\mu^g$ . By construction, this set  $\{J_v^g : v \in W_\delta\}$  contains an orbit of  $\{\phi_t^g\}$ . Since  $W_\delta$  is simply connected, this orbit contracts in  $\{J_v^g : v \in W_\delta\}$  and hence in  $\mathcal{A}_\mu^g$ .

It remains to construct the family  $(J_w, C_w)$  on  $S^2 \times S^2$  for  $w \in D \subset \mathbb{C}$ . To do this, take two copies of  $\mathbb{C} \times \mathbb{CP}^1$  and glue them via the map

$$((\mathbb{C} - \{0\}) \times \mathbb{CP}^1)_1 \rightarrow ((\mathbb{C} - \{0\}) \times \mathbb{CP}^1)_2 : (z, [v_0 : v_1])_1 \mapsto (\frac{1}{z}, [z^2 v_0 + w z v_1 : v_1])_2.$$

Identify the resulting complex manifold  $M_w$  with  $(S^2 \times S^2, J_w)$  so as to preserve the product structure near  $F_* = (\{0\} \times \mathbb{CP}^1)_2$ . For each  $w \neq 0$  and  $a \in \mathbb{C}$ , there is a holomorphic section  $Z_{a,w}$  of  $M_w$  containing the points

$$(z, [1 : a - \frac{z}{w}])_1, \quad (u, [wa : au - \frac{1}{w}])_2,$$

where  $u = 1/z$ . As  $a$  varies (and  $w$  is fixed) these are disjoint. Hence they must lie in class  $A$ . Moreover  $C_w = Z_{0,w}$  contains the points  $(u, [0 : 1])_2$  and so its image in  $S^2 \times S^2$  is flat near  $F_*$ . Finally, if we make  $\phi_t^0$  act on each copy of  $\mathbb{C} \times \mathbb{CP}^1$  via  $(z, [v_0 : v_1])_j \mapsto (z, [v_0 : e^{-it}v_1])_j$  and then it takes  $M_w$  to  $M_{e^{it}w}$ . The other properties are obvious.  $\square$

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<sup>7</sup>Note that we need  $g = 1$  here in order for the problem to have nonnegative index: when  $g > 1$  generic elements in  $\mathcal{A}_\mu^g$  do not admit curves in class  $A$ .

**Corollary 5.9** *For all  $i$  and  $g > 0$ , the kernel of the map*

$$\pi_i(\text{Diff}_0(\Sigma \times S^2)) \otimes \mathbb{Q} \rightarrow \pi_i(\mathcal{S}_\mu) \otimes \mathbb{Q}$$

*is independent of  $\mathbb{Q}$ .*

**Proof:** The kernel of this map is the image of  $\pi_i(G_\mu^g)$ . The result follows from the previous lemma because the inclusion  $G_\mu^g \rightarrow \text{Diff}_0(M)$  factors through  $\mathcal{D}_0$ .  $\square$

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